Quantum Cloning and Distributed Measurements

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We study measurements on various subsystems of the output of a universal $1 \to 2$ cloning machine, and establish a correspondence between these measurements at the output and effective measurements on the original input. We show that one can implement sharp effective measurement elements by measuring only two out of the three output systems. Additionally, certain complete sets of sharp measurements on the input can be realised by measurements on the two clones. Furthermore, we introduce a scheme that allows to restore the original input in one of the output bits, by using measurements and classical communication – a protocol that resembles teleportation.

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I. INTRODUCTION

The No-Cloning theorem [1] states that laws of quantum mechanics forbid to design an apparatus which is always successful in making an exact copy of an unknown quantum state. The fact that it is possible to make either imperfect copies with probability one [2] or perfect copies with probability less than one [3] is by now well established, and upper bounds for these scenarios have been derived [4–6].

So far this topic has been mainly addressed with the purpose to study a fundamental concept of quantum mechanics and its implication in connection with quantum information. Neither has a cloning transformation been realised experimentally so far, nor has cloning been shown to be a useful concept for quantum information processing. Recently, though, an experimental scheme for the realisation of an optimal cloning process via stimulated emission has been suggested [7]. Furthermore, imperfect cloning was shown to enhance the efficiency of imperfect detectors [8], and to improve the performance of some quantum computation tasks [9].

A quantum cloning transformation spreads the information that is contained in the input state over the entangled wave function of the output state. In this article we study various measurements at the output of an optimal universal $1 \to 2$ quantum cloning device, i.e. a black box that takes one unknown quantum bit as input and creates three entangled outputs. Two of those subsystems are the clones, one is an auxiliary system. We relate the measurements on the output systems to "effective" measurements at the input.

When having access to the total output state, applying the inverse cloning transformation evidently leads to the original input state and thus gives access to the original information. Let us imagine now that one has access only to a part of the output, e.g. if one of the subsystems was lost. Can one still make a measurement at the output that corresponds to a sharp measurement at the input? Our counterintuitive result is that this is indeed the case, when one has access to the two clones only. It turns out that the sharp measurements which can be implemented in this way are capable to extract the *maximum* accessible Shannon information on the – uniformly di tributed – input state. On the other hand, by measurements on the ancilla one can also access *some* Shannon information on the input state. (The quantitative calculation is an adaption of that given in [10].) In this sense the information contained in the input is spread in a partly redundant way over the output of the cloner. This property might be of use in potential quantum error correction schemes based on quantum cloning.

This paper aims at a systematic study of the distribution of information at the cloning output by investigating general measurements of various combinations of subsystems of the output. Questions regarding the mutual information for one-particle subsystems only at the cloning output have been addressed in [11]. Here we also study the effect of the entanglement at the output and derive explicit expressions for effective measurements on the input that correspond to measurements performed on subsystems of the output.

A further motivation for our study is related to the present search for better understanding of multiparticle entanglement: the cloning output is a special three-party entangled state. We will show that its peculiar properties allow to restore the original state in one of the output qubits by performing certain measurements on the other qubits and communicating classically – a protocol that bears a likeness to teleportation.

Note that one cannot find an analogous state for a two qubit system by studying the output of a universal $1 \to 1$ cloner [12]: the fidelity $F = 1$ is reached by applying the identity, and achieving a fidelity smaller than one requires an ancilla with a dimension higher than two.

Our work is organised as follows: in section II we remind the reader of the optimal $1 \to 2$ cloning transformation. In section III we present measurements of various subsystems of the cloning output and compare with the according effective measurements at the input. Thus we show that it is enough to possess two subsys-

tems of the output in order to perform measurements that correspond to sharp measurements at the input. In section IV we introduce the possibility of restoring the input in a subsystem of the output state by performing a measurement on a different output subsystem and then communicating classically. Finally, we give a summary.

II. REVISITING THE UNIVERSAL $1 \rightarrow 2$ CLONER

The universal $1 \rightarrow 2$ cloner in two dimensions takes as input a general pure input state $|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle$, the density matrix of which will be written as

$$|\varphi\rangle\langle\varphi| = \rho_{in} = \frac{1}{2}(\mathbb{1} + \vec{s}_{in}\cdot\vec{\sigma}) \tag{1}$$

where $\vec{\sigma} = \{\sigma_x, \sigma_y, \sigma_z\}$ are the Pauli matrices and $\vec{s}_{in} = \{s_x, s_y, s_z\}$ defines the input Bloch vector with unit length.

A cloning transformation U is specified through its action on the basis states. The family of optimal universal $1 \rightarrow 2$ cloning transformations is given in [13]. A special choice of phases and the ancilla leads to the transformation which we will use throughout this paper, namely

$$U|0\rangle|0\rangle|0\rangle = \sqrt{\frac{2}{3}}|00\rangle|1\rangle - \frac{1}{\sqrt{6}}(|01\rangle + |10\rangle)|0\rangle \ , \tag{2}$$

$$U|1\rangle|0\rangle|0\rangle = -\sqrt{\frac{2}{3}}|11\rangle|0\rangle + \frac{1}{\sqrt{6}}(|10\rangle + |01\rangle)|1\rangle \ , \tag{3}$$

where the first and second bit on the right hand side refer to the clones, and the third bit is the ancilla. Note that this transformation differs in the ancilla from the one considered in [2]. Our choice is such that the Bloch vector of the ancilla's reduced density matrix is given by $\vec{s}^{a} = -\frac{1}{3}\vec{s}_{in}$. This results in a special symmetry of the cloning transformation which we will introduce later.

We will recall further properties of this transformation in the next sections, wherever they are needed.

The connection between optimal quantum cloning and optimal state estimation has been established in [5] for the case of infinitely many copies. In our paper we are interested in measurements at the output of an optimal cloner for the case of two copies only.

III. MEASUREMENTS ON SUBSYSTEMS OF THE OUTPUT

A. Effective POVM's

The aim of this section is to relate possible measurements on the input ρ_{in} to measurements on parts of the output of the cloner, namely on one clone, or the ancilla, or the ancilla and one clone, or on two clones. For this purpose we will introduce the notion of *effective* POVM's.

A general measurement or *positive operator valued measure* (POVM) [10] on a quantum state ρ is described by a set of operators $\{F_i\}$ which is a resolution of the identity,

$$\sum_i^n F_i = \mathbb{1} \ , \tag{4}$$

where each *POVM element* F_i is a positive operator and is associated with one measurement outcome such that the probability of occurrence of this outcome is given by $p_i = \mathrm{Tr}(F_i\rho)$.

The cloning transformation connects the input and the output state unitarily. Two of the input qubits (blank and auxiliary qubit) are in a known prescribed state. Any measurement defined by a POVM on the output system ρ_{out} can be alternatively described by an *effective* POVM on the input state ρ_{in}, using the equality

$$\begin{aligned} \mathrm{Tr}[F_i\rho_{out}] &= \mathrm{Tr}[F_i U\rho_{in}\otimes|00\rangle_{ba}\langle 00|U^\dagger] = \\ &= Tr[{}_{ba}\langle 00|U^\dagger F_i U|00\rangle_{ba}\,\rho_{in}] \\ &= \mathrm{Tr}[E_i\rho_{in}] \end{aligned} \tag{5}$$

where $|00\rangle_{ba}$ denotes the blank and the ancilla input. Thus we have found for any POVM element F_i on the output Hilbert space the corresponding *effective POVM element* E_i on the input as

$$E_i = {}_{ba}\langle 00|U^\dagger F_i U|00\rangle_{ba} \,. \tag{6}$$

In the following subsections we proceed to calculate the effective POVM elements E_i for measurements which act on certain subsystems of the output, for example POVM elements $F_i^c\otimes\mathbb{1}^c\otimes\mathbb{1}^a$ which are measurements on just one clone.

Which are the effective measurements on the input that correspond to "valid" measurements on the output? These effective POVM's have to satisfy two criteria: (a) each element of the effective POVM must be realizable by a corresponding POVM element on the output and (b) the collection of corresponding POVM elements at the output must be complete, i.e. they have to add up to the identity. As we will see, we can establish this correspondence for several POVM's on certain subsystems of the output.

Our main interest lies in sharp POVM elements, that is operators of rank one. The occurrence of the result of a sharp measurement allows to exclude states orthogonal to the support of its corresponding matrix. Complete POVM's formed by sharp measurement elements play an important role in extracting information about the input state. It was already shown by Davies [14] that one can always optimize the accessible Shannon information by a POVM with sharp elements. If the set of input states has uniform distribution of the Bloch vector over the unit sphere, then actually *any* POVM formed only by sharp elements optimizes the accessible Shannon information. This is due to isotropy, since in this case the occurrence

of each measurement outcome reveals the same amount of information about the input state. This statement can also be checked explicitly by calculations following those given in [10]. Moreover, optimal state estimation for isotropically distributed one-qubit states [15] can be realised by any sharp POVM. This can be seen from work by Vidal et al. [16].

B. General properties of measurements on subsystems

In the following, we will investigate in which situation it is possible to perform a sharp measurement on the input system by measuring only a subsystem of the output of the $1 \to 2$ cloner, for example by measuring the two clones, but not the auxiliary system. In this investigation the following theorem is useful, which states that for each output POVM element belonging to this class the conditional density matrix of the unmeasured system must necessarily be in a state independent of the input state.

Theorem 1 *Consider a product Hilbert space of system A and B prepared in a product state described by a density matrix $\rho_A \otimes \rho_B$. Let U be a unitary operator that maps the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ onto a Hilbert space of the same dimension with a product structure $\mathcal{H}_C \otimes \mathcal{H}_D$. Given the POVM element F that acts on system C, the corresponding effective POVM element E on system A is described by*

$$E = Tr_B \left(\rho_B U^\dagger F \otimes \mathbb{1}_D U \right) \ . \tag{7}$$

If the operator E on $\mathcal{H}_A$ is of rank one (sharp POVM element), then the state of system D, conditioned on the measurement outcome at C, is independent of ρ_A.

The statement of the theorem can be rephrased by viewing this procedure as a quantum channel which maps the input state ρ_A onto the outgoing subsystem $\rho_D = \mathrm{Tr}_C \left(U \rho_A \otimes \rho_B U^\dagger \right)$ which is *not* measured. This mapping is completely positive (CP), and can be written in terms of the Kraus operators $\{A_i\}$,

$$\rho_A \to \rho_D = \sum_{i=1}^{N} A_i \rho_A A_i^\dagger \ , \tag{8}$$

where $\sum_{i=1}^{N} A_i^\dagger A_i = \mathbb{1}$. If we consider only one measurement outcome, as in the theorem, then the *conditional* state corresponding to this outcome is given by

$$\rho_D^{cond} = \frac{1}{M} \sum_{i \in K} A_i \rho_A A_i^\dagger \ , \tag{9}$$

where M is a normalization constant, and K is some subset of the index set $i = \{1, \dots N\}$. In the most simple case it will contain only one element.

The effective POVM element corresponding to this measurement outcome is therefore given by the Kraus operators as

$$E = \sum_{i \in K} A_i^\dagger A_i \ . \tag{10}$$

If the operator E is of rank 1, it can be written as $E = p \left|\tilde{\chi}\right\rangle \left\langle\tilde{\chi}\right|$ with some p satisfying $0 \le p \le 1$. Since any operator $A_i^\dagger A_i$ is positive, the structure of Eqn. (10) implies that each operator $A_i^\dagger A_i$ is of rank 1. We make a singular value decomposition ansatz $A_i = V S \tilde{V}$, where S is a positive, real diagonal matrix, and $V, \tilde{V}$ are unitary matrices. It follows that S must be of rank 1. Therefore, we can write each Kraus operator as $A_i = \gamma_i \left|\chi_i\right\rangle \left\langle\tilde{\chi}\right|$ with γ_i real and positive and $\sum \gamma_i^2 = p$, where the vectors $\left|\chi_i\right\rangle$ might be different for the various contributing Kraus operators, while the vector $\left|\tilde{\chi}\right\rangle$ is fixed by E. This immediately gives us the final state

$$\rho_D^{cond} = \frac{1}{M} \left\langle\tilde{\chi}\right| \rho_A \left|\tilde{\chi}\right\rangle \sum_{i \in K} \gamma_i^2 \left|\chi_i\right\rangle \left\langle\chi_i\right| \tag{11}$$

$$= \frac{1}{M'} \sum_{i \in K} \gamma_i^2 \left|\chi_i\right\rangle \left\langle\chi_i\right| \ . \tag{12}$$

Here M and M' are normalization factors. The final conditional state does no longer depend on the input state. This proves our theorem.

C. Measurements on one clone or on the ancilla

The most general POVM element on one single qubit of the output is given by the expansion of a hermitian matrix in terms of the Pauli matrices, namely

$$F_i = b_i \left(\mathbb{1} + \vec{f}_i \cdot \vec{\sigma} \right) \tag{13}$$

where $0 < b_i \le 1$ and $|\vec{f}_i| \le 1$ for positivity. (A sharp POVM element is characterized by $|\vec{f}_i| = 1$.) The corresponding effective POVM element at the input can be written in the same form as

$$E_i = a_i \left(\mathbb{1} + \vec{e}_i \cdot \vec{\sigma} \right) \ . \tag{14}$$

In order to find the parameters a_i and $\vec{e}_i$ as functions of b_i and $\vec{f}_i$ we use the equality

$$\mathrm{Tr}(E_i \rho_{in}) = \mathrm{Tr}(F_i \rho_{out,red}) \ , \tag{15}$$

where $\rho_{out,red}$ refers to the according subsystem at the output. We find immediately

$$a_i \left(1 + \vec{s}_{in} \cdot \vec{e}_i\right) = b_i (1 + \vec{s}_{out,red} \cdot \vec{f}_i^{\,c}) \ . \tag{16}$$

Here $\vec{s}_{out,red}$ denotes the Bloch vector of the output state. For the universal cloner the Bloch vectors of both clones

have the same orientation as for the input [13]. The Bloch vector of the ancilla can be chosen to be antiparallel to that [17]. After applying the optimal transformation the Bloch vector for a clone is shrunk by a factor of $2/3$ with respect to the input, and we denote it as $\vec{s}^{c} = 2/3 \cdot \vec{s}_{in}$. The Bloch vector of the ancilla is here given by $\vec{s}^{a} = -1/3 \cdot \vec{s}_{in}$, according to the transformation U in equation (2).

Since equation (16) has to hold for all input states and therefore for all Bloch vectors $\vec{s}_{in}$, we find for the parameters of measurements on one clone, denoted by the superscript c:

$$a_i^c = b_i^c \ , \tag{17}$$

$$\vec{e}_i^{\,c} = \frac{2}{3}\vec{f}_i^{\,c} \tag{18}$$

Thus only effective POVM elements with $|\vec{e}_i^{\,c}| \leq 2/3$ can be realized by a measurement on one clone alone; sharp measurements like projection measurements are excluded. This is consistent with the fact that the clone can be understood as a mixture of a completely random state with probability $\frac{1}{3}$ and the input state with probability $\frac{2}{3}$. So, the result of passing our original state through a Stern-Gerlach apparatus which fails with probability $\frac{1}{3}$ would be the same as the one obtained by running one clone through a perfect Stern-Gerlach apparatus.

An analogous result is obtained for measurements on the ancilla. We label the parameters of the POVM element in equation (13) with a superscript a for this case and find

$$a_i^a = b_i^a \ , \tag{19}$$

$$\vec{e}_i^{\,a} = -\frac{1}{3}\vec{f}_i^{\,a} \ . \tag{20}$$

As expected, measuring the ancilla alone cannot correspond to making sharp measurements on the input.

We infer that the prescription of a complete POVM on the input system defines a unique and valid POVM on one clone (or on the ancilla), provided that for each POVM element the relation $|\vec{e}_i^{\,c}| \leq 2/3$ (or $|\vec{e}_i^{\,a}| \leq \frac{1}{3}$) holds. The completeness relation for measurements at the input reads $\sum_i a_i = 1$ and $\sum_i a_i \vec{e}_i = 0$. If these equations hold, the completeness relation for measurements at the output, namely $\sum_i b_i = 1$ and $\sum_i b_i \vec{f}_i = 0$, is here automatically satisfied, due to the equality $a_i = b_i$ and the fact that $\vec{e}_i$ and $\vec{f}_i$ are related by a constant factor.

D. Coherent measurements on two subsystems

In the previous section we investigated, which effective POVMs can be realized by measurements on single subsystems of the output of the $1 \to 2$ cloner. In this section we consider coherent measurements on two subsystems.

Before studying these specific measurements, let us first introduce general measurements on higher-dimensional spaces: any POVM element acting in a d-dimensional Hilbert space can be written in the form

$$F_i = b_i \left(\mathbb{1} + \vec{f}_i \cdot \vec{\tau} \right) \tag{21}$$

where the parameters b_i and $\vec{f}_i$ are real, and $\vec{\tau}$ denotes the $d^2 - 1$ generators of SU(d), which obey $\mathrm{Tr}(\tau_i \tau_j) = 2\delta_{ij}$. Which are the necessary conditions, such that equation (21) describes a valid POVM element? We will only give two obvious conditions. A complete and explicit characterisation of necessary *and* sufficient conditions is beyond the scope of this paper. As the eigenvalues of each POVM element have to be in the interval $[0, 1]$, the dimension d of the underlying Hilbert space leads to the constraint $0 \leq \mathrm{Tr}\,(F_i) \leq d$, and therefore to $0 \leq b_i \leq 1$.

Another necessary condition is $|\vec{f}_i|^2 \leq d(d-1)/2$, for the following reason: a projection measurement can be written in the form $F_i = |\phi\rangle\langle\phi| = \frac{1}{d}\left(\mathbb{1} + \vec{f}_i \cdot \vec{\tau}\right)$, and the condition $\mathrm{Tr}\left(F_i^2\right) = 1$ then leads to $|\vec{f}_i|^2 = d(d-1)/2$. Since a general POVM element is proportional to a convex combination of projectors, we obtain in general $|\vec{f}_i|^2 \leq d(d-1)/2$.

1. Measurements on the two clones

Let us turn to measurements on the two clones. For our purposes it is convenient to rewrite the cloning transformation (2), using the Bell basis for the two clones:

$$U|0\rangle|0\rangle|0\rangle = \frac{1}{\sqrt{3}}\left|\phi^+\right\rangle|1\rangle + \frac{1}{\sqrt{3}}\left|\phi^-\right\rangle|1\rangle - \frac{1}{\sqrt{3}}\left|\psi^+\right\rangle|0\rangle \ , \tag{22}$$

$$U|1\rangle|0\rangle|0\rangle = -\frac{1}{\sqrt{3}}\left|\phi^+\right\rangle|0\rangle + \frac{1}{\sqrt{3}}\left|\phi^-\right\rangle|0\rangle + \frac{1}{\sqrt{3}}\left|\psi^+\right\rangle|1\rangle \ , \tag{23}$$

where the Bell states are defined as

$$\left|\phi^\pm\right\rangle = 1/\sqrt{2}\left(|00\rangle \pm |11\rangle\right) \ , \tag{24}$$

$$\left|\psi^\pm\right\rangle = 1/\sqrt{2}\left(|01\rangle \pm |10\rangle\right) \ . \tag{25}$$

Only the symmetric Bell states appear in the cloning transformation, because the two clones are required to be in the symmetric subspace [6]. Their reduced density matrix can be written in a simple form in the Bell basis $\{\phi^+, \psi^+, \phi^-, \psi^-\}$:

$$\rho^{cc} = \mathrm{Tr}_a \varrho_{out} = \frac{1}{3}\begin{pmatrix} 1 & s_x & s_z & 0 \\ s_x & 1 & is_y & 0 \\ s_z & -is_y & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{26}$$

where the trace is performed over the ancilla.

As this density matrix has only support in the 3-dimensional symmetric subspace, we can decompose it

using the eight generators of SU(3), namely λ_k (which are labelled in a standard way, see e.g. [18]),

$$\rho^{cc} = \frac{1}{3}\left(\mathbb{1} + \begin{pmatrix} s_x \\ -s_y \\ s_z \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_7 \\ \lambda_4 \end{pmatrix}\right) = \frac{1}{3}\left(\mathbb{1} + \vec{s}^{cc}\vec{\lambda}\right) \quad . \tag{27}$$

Note that the eight-dimensional generalized Bloch vector $\vec{s}^{cc}$ has only three non-vanishing entries.

For measurements on the two clones it is sufficient to consider POVM's on the symmetric subspace of two qubits. Any such POVM element can be written in the form

$$F_i^{cc} = b_i^{cc}\left(\mathbb{1} + \vec{f}_i^{cc} \cdot \vec{\lambda}\right) \tag{28}$$

where the parameters b_i^{cc} and $\vec{f}_i^{cc}$ are real, with the constraints $0 \leq b_i^{cc} \leq 1$, and $|\vec{f}_i^{cc}|^2 \leq 3$, as explained below equation (21). Any anti-symmetric component of a measurement on the two clones has no effect on the effective POVM.

As in the previous subsection we can now relate the parameters of an effective POVM element on the input Hilbert space,

$$E^{cc} = a_i^{cc}\left(\mathbb{1} + \vec{e}_i^{cc} \cdot \vec{\sigma}\right) \ , \tag{29}$$

to the parameters in equation (28) via the equality

$$Tr[E_i^{cc}\rho_{in}] = Tr[F_i^{cc}\rho^{cc}] \ , \tag{30}$$

which leads to the assignment

$$\begin{aligned} a_i^{cc} &= b_i^{cc} \ , \\ \vec{e}_i^{cc} &= \frac{2}{3}\begin{pmatrix} f_1^{cc} \\ -f_7^{cc} \\ f_4^{cc} \end{pmatrix}_i \quad . \end{aligned} \tag{31}$$

Note that one can show that this relation can be expressed more simple as

$$E_i^{cc} = \frac{2}{3} Tr_c\left(F_i^{cc}\right) \tag{32}$$

where the trace is performed over one of the clones and the resulting operator is interpreted as acting on the input state.

Given the output POVM element F_i, this assignment uniquely determines the effective POVM element E_i on the input. However, we see that conversely the prescription of the effective POVM E_i leaves in general five parameters of the output POVM element F_i undetermined. In order to establish whether a particular effective POVM element can be realised by a measurement on the two clones, one needs to show that there exists a choice of the undetermined parameters f_ν^{cc} with $\nu = 2, 3, 5, 6, 8$, such that the resulting operators F_ν^{cc} are valid POVM elements.

Here a complete POVM on the input does not necessarily correspond to a complete POVM at the output. In some cases, though, choosing the free parameters for the output POVM elements appropriately will allow the output POVM to fulfill the completeness relation.

We will now derive a complete set of sharp measurements on the two clones which corresponds to rank 1 effective POVM elements on the input. The derivation uses a symmetry argument and the general statement about sharp measurements given in section III B.

The optimal universal $1 \rightarrow 2$ cloning transformation U obeys the following symmetry for any matrix $V \in SU(2)$:

$$U\left(V \otimes \mathbb{1} \otimes \mathbb{1}\right)|000\rangle = \left(V \otimes V \otimes V\right) U\,|000\rangle \ . \tag{33}$$

According to section III B the sharp measurement on the two clones has to leave the ancilla in a state independent of the input. The symmetry property (33) allows us to consider a fixed state of the ancilla, and we choose the state $|0\rangle$.

Given a general input state $|\varphi\rangle$, inspection of the cloning transformation (2) leads to the only choice for a symmetric projection measurement on the clones that leaves the ancilla in the state $|0\rangle$, namely $F_i^{cc} = p\,|11\rangle\,\langle 11|$ with $0 \leq p \leq 1$. Using equation (32) it is straightforward to calculate the corresponding effective POVM element, which is given by $E_i^{cc} = \frac{2}{3}p\,|1\rangle\langle 1|$.

Making use of the symmetry of the universal cloner we find that the corresponding complete class of POVM elements on the symmetric subspace of the two clones leading to sharp effective POVM elements on the input is given by

$$F_i^{cc} = p\,|\chi_i\rangle\langle\chi_i| \otimes |\chi_i\rangle\langle\chi_i| \otimes \mathbb{1}_a \ . \tag{34}$$

The corresponding sharp effective POVM elements on the input are

$$E_i^{cc} = \frac{2}{3}p\,|\chi_i\rangle\langle\chi_i| \tag{35}$$

Clearly, we cannot implement a measurement like in a Stern Gerlach apparatus, since this would correspond to $p = 3/2$. However, one can e.g. choose a mixture of three set-ups, with equal probability, discriminating in the x, y, and z-direction. The complete set of effective POVM elements is then given by

$$\{E_i^{cc}\} = \{\frac{1}{3}P_{x+}, \frac{1}{3}P_{x-}, \frac{1}{3}P_{y+}, \frac{1}{3}P_{y-}, \frac{1}{3}P_{z+}, \frac{1}{3}P_{z-}\} \ , \tag{36}$$

which corresponds to the complete set of POVM elements on the two clones

$$\begin{aligned} \{F_i^{cc}\} = \{&\frac{1}{2}\,|x+\rangle\,|x+\rangle\,\langle x+|\,\langle x+|\,, \frac{1}{2}\,|x-\rangle\,|x-\rangle\,\langle x-|\,\langle x-|\,, \\ &\frac{1}{2}\,|y+\rangle\,|y+\rangle\,\langle y+|\,\langle y+|\,, \frac{1}{2}\,|y-\rangle\,|y-\rangle\,\langle y-|\,\langle y-|\,, \\ &\frac{1}{2}\,|z+\rangle\,|z+\rangle\,\langle z+|\,\langle z+|\,, \frac{1}{2}\,|z-\rangle\,|z-\rangle\,\langle z-|\,\langle z-|\} \ . \end{aligned} \tag{37}$$

Another possibility is given by the effective POVM

$$\{E_i^{cc}\} = \{\frac{1}{2}\left|\vec{n}_1\right\rangle\left\langle\vec{n}_1\right|, \frac{1}{2}\left|\vec{n}_2\right\rangle\left\langle\vec{n}_2\right|, \frac{1}{2}\left|\vec{n}_3\right\rangle\left\langle\vec{n}_3\right|, \frac{1}{2}\left|\vec{n}_4\right\rangle\left\langle\vec{n}_4\right|\}\ , \tag{38}$$

where the Bloch vectors $\vec{n}_i$ point to the corners of a regular tetrahedron. This corresponds to measurements at the output

$$\{F_i^{cc}\} = \{\frac{3}{4}\left|\vec{n}_1\vec{n}_1\right\rangle\left\langle\vec{n}_1\vec{n}_1\right|, \dots \frac{3}{4}\left|\vec{n}_4\vec{n}_4\right\rangle\left\langle\vec{n}_4\vec{n}_4\right|\}\ . \tag{39}$$

This measurement corresponds to optimal state estimation [15] on two identical qubits that have uniform a-priori probability distribution.

Note that the POVM elements (34) are products of POVM elements on each individual clone. Therefore, it is possible to implement by means of local operations and classical communication (LOCC) a POVM on the two clones which contains *some* sharp effective POVM elements. However, it can be seen that a complete set of such elements as in (37) or (39) cannot be realised by LOCC measurements. The reason for this is that any complete POVM realised by LOCC operations has an antisymmetric component. Extending or completing our POVMs (which were restricted to the symmetric space for convenience) will always lead to non-local features.

In this section we have found the surprising result that sharp effective measurements on the input can be achieved when measuring the two clones only, i.e. the information that is contained in the ancilla is in this sense redundant. The examples of effective POVM's we presented are actually suited to optimise the fidelity of state estimation or the accessible Shannon information for a uniformly distributed input qubit. These tasks can therefore be performed equally well with the input state or the two clones alone, see [5] and [16].

2. Measurements on one clone and the ancilla

The reduced density matrix of the two subsystems formed by one clone and one ancilla acquires the following form if expressed in the Bell basis:

$$\rho^{ca} = \frac{1}{12}\begin{pmatrix} 1 & s_x & s_z & 3is_y \\ s_x & 1 & is_y & 3s_z \\ s_z & -is_y & 1 & -3s_x \\ -3is_y & 3s_z & -3s_x & 9 \end{pmatrix}\ , \tag{40}$$

which does have support on the antisymmetric space, i.e. the entries for $\left|\psi^-\right\rangle$ (last row and column) do not vanish.

However, comparing (40) with the density matrices of the two clones (26), we note that the symmetric part for the clone-ancilla system is equal to the density matrix of the two clones times the factor $\frac{1}{4}$. Consequently, all sharp effective POVM elements E_i corresponding to measurements F_i on the clone-clone system can be realised by the same measurements F_i on the clone-ancilla system, although the weight a_i in the effective POVM E_i will be decreased by a factor 1/4.

As in the previous paragraph about measurements on two clones one can also derive a correspondence like in (31), where the vector $\vec{f}$ will now be 15-dimensional, as the general POVM has to be expanded in terms of the SU(4) generators [18]. We give this correspondence for completeness:

$$\begin{aligned} a_i^{ca} &= b_i^{ca}(1-\sqrt{\frac{2}{3}}f_{15}^{ca})\ , \\ \vec{e}_i^{ca} &= \frac{1}{6(1-\sqrt{\frac{2}{3}}f_{15}^{ca})}\begin{pmatrix} f_1^{ca} - 3f_{13}^{ca} \\ -f_7^{ca} - 3f_{10}^{ca} \\ f_4^{ca} + 3f_{11}^{ca} \end{pmatrix}_i\ . \end{aligned} \tag{41}$$

Can we find a complete measurement on a the clone-ancilla subsystem of the output that corresponds to sharp measurements at the input, like in the previous subsection?

We follow the same arguments as in the case of measurements on the two clones. First, we rewrite the cloning transformation (2), now sorting the terms according to the value of the first qubit:

$$U\left|000\right\rangle = \frac{1}{\sqrt{12}}\left(\left|0\right\rangle\left(\left|\psi^+\right\rangle + 3\left|\psi^-\right\rangle\right) - \left|1\right\rangle\left(\left|\phi^+\right\rangle + \left|\phi^-\right\rangle\right)\right)\ , \tag{42}$$

$$U\left|100\right\rangle = \frac{1}{\sqrt{12}}\left(-\left|1\right\rangle\left(\left|\psi^+\right\rangle - 3\left|\psi^-\right\rangle\right) + \left|0\right\rangle\left(\left|\phi^+\right\rangle - \left|\phi^-\right\rangle\right)\right)\ . \tag{43}$$

By demanding (compare the argument in the previous section) that the first clone should be in state $\left|0\right\rangle$ after a projection measurement, we find that the measurement $F_i = \left|\kappa_{F_i}\right\rangle\left\langle\kappa_{F_i}\right|$ on the other clone and the ancilla has to be given by

$$\left|\kappa_{F_i}\right\rangle = A\left(3\left|\psi^+\right\rangle + \left|\psi^-\right\rangle\right) + B\left(\left|\phi^+\right\rangle - \left|\phi^-\right\rangle\right) \tag{44}$$

where A and B are free parameters.

Using (6) it is straightforward to check that the corresponding effective POVM is indeed of rank one, i.e. $E_i = \left|\kappa_{E_i}\right\rangle\left\langle\kappa_{E_i}\right|$, and is given by

$$\left|\kappa_{E_i}\right\rangle = \frac{1}{\sqrt{12}}\left(6A\left|0\right\rangle + 2B\left|1\right\rangle\right)\ . \tag{45}$$

Can we find a complete POVM at the output, i.e. a set $\{F_i\}$ with $\sum_i F_i = \mathbb{1}$, that corresponds to sharp POVM elements on the input? Given the symmetry of the cloning transformation we know that all the POVM elements F_i on one clone and ancilla must be projectors onto states of the form $V\otimes V\left|\kappa_{F_i}\right\rangle$, with $\left|\kappa_{F_i}\right\rangle$ given in equation (44). The antisymmetric subspace is invariant under the unitary transformation $V\otimes V$, and a symmetric state remains symmetric after applying $V\otimes V$. Therefore the weight of the symmetric subspace in *any* projector onto $V\otimes V\left|\kappa_{F_i}\right\rangle$ is at least 9 times as much as the weight of the antisymmetric subspace, due to the

term proportional to A in (44). A sum of such projectors cannot be a resolution of the identity, as the weight of the symmetric subspace in the identity is only three times as much as for the antisymmetric subspace.

Thus we cannot find a complete POVM acting on one clone and the ancilla that corresponds to sharp POVM elements on the input, despite the fact that we can realize individual sharp POVM elements. Actually, some of these individual sharp effective POVM elements can be realised by local (LOCC) means since the states (44) contain product states, for example for $A = 0$.

IV. RESTORATION VIA LOCAL MEASUREMENTS AND COMMUNICATION

In the previous section we investigated which measurements are possible if one has access only to some subsystem of the output of the $1 \to 2$ universal cloner, instead of having access to the whole output or to the original input state. In this section we will study the situation in which the output subsystems are distributed to several parties to whom we refer as Alice, Bob and, in some situations, Charlie. The parties are allowed to communicate classically with each other.

Actually, the situation with respect to effective POVM elements is very simple in this scenario where we have access to the complete output. Trivially, we can implement a complete set of effective sharp POVM's. This follows from the fact that *any* projection of the complete output onto a state $|\Psi_{out}\rangle$ results in a sharp effective measurement on the input system: following the formalism given in Eqns. (5) and (6) we find the effective POVM to be a projection onto the (unnormalized) state $\langle 00| U^\dagger |\Psi_{out}\rangle$ of the input system. Here $\langle 00|$ refers to the intitial state of blank and ancilla qubit. Any sharp POVM on the complete output system therefore leads immediately to a sharp effective POVM on the input system.

The interesting result shown in this part of the paper is that it is possible to recover the original state in one subsystem by a process resembling quantum teleportation [19]. Naturally, whenever restoration is possible, any effective POVM can be trivially implemented by applying the corresponding POVM to the restored qubit.

We will discuss different scenarios: Perfect restoration of the input is possible if one party holds two subsystems and a second party holds one subsystem, in which the original will be restored. Probabilistic restoration in the ancilla can be achieved when each subsystem is held by a different party.

To begin with, let us describe a general restoration process of the unknown input qubit in one subsystem of the output. In the first step, the subsystems, except the target system, are being measured and the measurement results are transmitted to the site of the target qubit. In the second step, the owner of the target system acts on it, for example by executing an unitary operation or by performing a generalized measurement. The desired result is that the target bit will be in the unknown state of the input bit. If this is possible for all measurement outcomes in both steps, then this process is deterministic, otherwise we speak of probabilistic restoration.

Both steps can be described by completely positive maps. The first step, described by Kraus operators A_i, map the input system onto the target qubit. The second step is described by Kraus operators $B_j^{(i)}$ which depend on the measurement result of the first step and map the state of the target qubit onto itself. In its full generality, the resulting density matrix of the target qubit, conditioned on measurement results in both steps, is given by

$$\tilde{\rho}_{out}^{cond} = \sum_{j \in L(K)} \sum_{i \in K} B_j^{(i)} A_i \rho_{in} A_i^\dagger {B_j^{(i)}}^\dagger . \qquad (46)$$

The index sets K and $L(K)$ arise since each measurement outcome in the two steps could be described in full generality by more than one Kraus operator. Each event which successfully restores the input state in the target qubit must satisfy the equation (for some positive p)

$$\sum_{j \in L(K)} \sum_{i \in K} B_j^{(i)} A_i \rho_{in} A_i^\dagger {B_j^{(i)}}^\dagger = p \rho_{in} \qquad (47)$$

for all states ρ_{in}, especially for pure states.

Pure states, however, are extreme points of the convex set of all states and can therefore not be written as convex sums of different density matrices. Consequently, we find the stronger condition

$$B_j^{(i)} A_i \rho_{in} A_i^\dagger {B_j^{(i)}}^\dagger = p_{ij} \rho_{in} \qquad (48)$$

which holds for all $j \in L(K)$ and $i \in K$ with some non-negative p_{ij} for all pure ρ_{in}. Due to linearity, equation (48) also has to hold for all mixed states ρ_{in}. Therefore we arrive at the condition $B_j^{(i)} A_i \propto \mathbb{1}$ for $p_{ij} \neq 0$, and $B_j^{(i)} A_i = 0$ for $p_{ij} = 0$. In conclusion, we find that

$$\sum_{j \in L(K)} \sum_{i \in K} A_i^\dagger {B_j^{(i)}}^\dagger B_j^{(i)} A_i = p \mathbb{1} . \qquad (49)$$

This means that, as a principle, the event of a successful restoration does not leak any kind of information about the restored state to the restoring parties. This statement cannot be reversed in general, but it can be used as a guidance to find procedures for restoration.

Following these general considerations, we can distinguish two different scenarios. Both are using a sharp measurement in the first step. In the first scenario, the operator A_i from the first step is proportional to a unitary operator. This means that no information about the input state is revealed, and we can choose that the second step consists of a unitary operation described by only one Kraus operator $B^{(i)} \sim A_i^\dagger$ [20]. If all Kraus operators fall into this scenario, we can in all events restore the input state in the target qubit.

In the second scenario, the operator A_i is not proportional to a unitary operator, and therefore the corresponding preliminary effective POVM element $A_i^\dagger A_i \not\sim \mathbb{1}$ reveals some information about the input state. However, this can be compensated, with some probability, by a measurement in the second step. The idea is that one Kraus operator of the second step 'compensates' the knowledge, and we obtain for the overall effective POVM element $A_i^\dagger B_j^{(i)\dagger} B_j^{(i)} A_i \sim \mathbb{1}$.

One can immediately determine the optimal choice of $B_j^{(i)}$, given A_i, by looking at the singular value decomposition of $A_i = U \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) V$ where U, V are unitary operators, and $1 \geq a \geq b \geq 0$ are constants. Then the choice $B_j^{(i)} = V^\dagger \left(\begin{smallmatrix} b/a & 0 \\ 0 & 1 \end{smallmatrix}\right) U^\dagger$ results in $B_j^{(i)} A_i = b\mathbb{1}$. This choice guarantees the restoration of the input state in the target qubit. Up to a pre-factor, the choice is unique. This pre-factor is constrained by the fact that the eigenvalues of $B_j^{(i)}$ may not exceed 1, and in our choice has been determined such that it maximises the probability of successful restoration. Note that a restoration is not possible if $b = 0$, corresponding to a sharp effective measurement in the first step. This is in agreement with Theorem 1 which states that in this case the target qubit is in a state independent of the input.

This formalism shows that any projection of the subsystems (except the target qubit) onto a pure state allows a probabilistic restoration unless this projection results in a sharp effective POVM element. The possibility of a probabilistic restoration is therefore not uncommon. We will now illustrate the deterministic and the probabilistic restoration in several settings.

A. Deterministic restoration in the ancilla or a clone

In this scenario Alice has coherent access to the two clones while Bob has only access to the ancilla. They are also able to do one-way classical communication from Alice to Bob. We now show that it is possible to recreate the original input state in the ancilla by a suitable measurement on the two clones, followed by a conditional unitary dynamics on the ancilla.

The output state of the cloner for an arbitrary input state is given by

$$
\begin{aligned}
U\left(\alpha\left|0\right\rangle+\beta\left|1\right\rangle\right)\left|0\right\rangle\left|0\right\rangle \;=\; & \tfrac{1}{\sqrt{3}}\left|\phi^{+}\right\rangle\left(-\beta\left|0\right\rangle+\alpha\left|1\right\rangle\right) \\
& +\tfrac{1}{\sqrt{3}}\left|\phi^{-}\right\rangle\left(\beta\left|0\right\rangle+\alpha\left|1\right\rangle\right) \\
& +\tfrac{1}{\sqrt{3}}\left|\psi^{+}\right\rangle\left(-\alpha\left|0\right\rangle+\beta\left|1\right\rangle\right) \quad (50)
\end{aligned}
$$

Note that a measurement in the Bell basis on the two clones corresponds to a measurement with effective POVM elements proportional to the identity operator (following Eqn. (32)), thereby revealing no information about the input state. Thus a Bell measurement on Alice's side allows the full restoration of the input state. The measurement will lead with equal probability of 1/3 to one of the three symmetric Bell states. Like in teleportation, Bob's remaining conditional state can then be transformed into the original input state by a suitable unitary operation, as shown in table I.

result	conditional state	unitary operation
$\left\|\phi^+\right\rangle$	$-\beta\left\|0\right\rangle+\alpha\left\|1\right\rangle$	$i\sigma_y$
$\left\|\phi^-\right\rangle$	$\beta\left\|0\right\rangle+\alpha\left\|1\right\rangle$	σ_x
$\left\|\psi^+\right\rangle$	$-\alpha\left\|0\right\rangle+\beta\left\|1\right\rangle$	$-\sigma_z$

TABLE I. A Bell measurement on the two clones leads to a conditional state of the ancilla, which can be rotated unitarily into the original input state.

In a changed scenario where Alice has control of one clone and the ancilla, and Bob has access to the other clone, Alice and Bob can restore again the original input state in Bob's clone by means of a Bell measurement of Alice and classical communication. To see this, we expand the output of the cloner for an arbitrary input in the Bell states of the combined system clone-ancilla:

$$
\begin{aligned}
U\left(\alpha\left|0\right\rangle+\beta\left|1\right\rangle\right)\left|0\right\rangle\left|0\right\rangle \;=\; & +\tfrac{1}{2}\sqrt{3}\left(\alpha\left|0\right\rangle+\beta\left|1\right\rangle\right)\left|\psi^{-}\right\rangle \\
& +\tfrac{1}{\sqrt{12}}\left(\alpha\left|0\right\rangle-\beta\left|1\right\rangle\right)\left|\psi^{+}\right\rangle \\
& +\tfrac{1}{\sqrt{12}}\left(\beta\left|0\right\rangle-\alpha\left|1\right\rangle\right)\left|\phi^{+}\right\rangle \\
& -\tfrac{1}{\sqrt{12}}\left(\beta\left|0\right\rangle+\alpha\left|1\right\rangle\right)\left|\phi^{-}\right\rangle \;. \quad (51)
\end{aligned}
$$

Alice performs a measurement in the Bell basis on her clone and the ancilla. With probability 3/4 she will obtain the result $\left|\psi^-\right\rangle$, and with probability 1/12 each one of the symmetric Bell states. She communicates the measurement result to Bob who performs the appropriate transformation (see table II) to restore the input state in his clone.

result	conditional state	unitary operation
$\left\|\phi^+\right\rangle$	$\beta\left\|0\right\rangle-\alpha\left\|1\right\rangle$	$-i\,\sigma_y$
$\left\|\phi^-\right\rangle$	$-\beta\left\|0\right\rangle-\alpha\left\|1\right\rangle$	$-\sigma_x$
$\left\|\psi^+\right\rangle$	$\alpha\left\|0\right\rangle-\beta\left\|1\right\rangle$	σ_z
$\left\|\psi^-\right\rangle$	$\alpha\left\|0\right\rangle+\beta\left\|1\right\rangle$	$\mathbb{1}$

TABLE II. A Bell measurement on one of the clones and the ancilla leads to a conditional state of the other clone, which can be rotated unitarily into the original input state.

Therefore we have shown that it is not necessary to perform the reverse cloning transformation on the total cloning output in order to recover the original: the three-particle entangled output has the property that measuring two subsystems and communicating classically restores the original either in a clone or in the auxiliary qubit. Thus in addition to the trivial way of restoration by revers cloning there is another way which does not require to operate on all output systems coherently.

B. Probabilistic restoration in the ancilla or a clone

Let us now look at the case where the three output subsystems are split up between three parties: Alice, Bob, and Charlie (who holds the ancilla). No coherent measurement on two subsystems are possible now. The parties wish to restore the original input state on Charlie's side by local measurements and classical communication. We will show that this is possible, although with probability smaller than 1.

The cloning output for an arbitrary state can be written as

$$\begin{aligned} &U(\alpha|0\rangle+\beta|1\rangle)|0\rangle|0\rangle= \\ &\quad\sqrt{\tfrac{2}{3}}\alpha|00\rangle|1\rangle+\tfrac{1}{\sqrt{6}}|01\rangle(-\alpha|0\rangle+\beta|1\rangle) \\ &\quad-\sqrt{\tfrac{2}{3}}\beta|11\rangle|0\rangle+\tfrac{1}{\sqrt{6}}|10\rangle(-\alpha|0\rangle+\beta|1\rangle)\ . \end{aligned} \tag{52}$$

Alice and Bob measure their clones in the basis $\{0,1\}$. They communicate their result to Charlie. If they find different outcomes, Charlie applies $-\sigma_z$, and the input system is restored in his ancilla. Otherwise, the corresponding effective POVM element turns out to be sharp, so that the auxiliary system is in a state independent of the input, and no recovery is possible. However, the three parties know which situation occurred. The restoration is successful with probability 1/3.

We show now that it is also possible to restore the input state in a clone held by Alice, while the other clone and the auxiliary system are held by Bob and Charlie, respectively. We reorder the cloning transformation

$$\begin{aligned} &U(\alpha|0\rangle+\beta|1\rangle)|0\rangle|0\rangle= \\ &\quad-\tfrac{1}{\sqrt{6}}\alpha|1\rangle|00\rangle+\left(\sqrt{\tfrac{2}{3}}\alpha|0\rangle+\tfrac{1}{\sqrt{6}}\beta|1\rangle\right)|01\rangle \\ &\quad+\left(-\tfrac{1}{\sqrt{6}}\alpha|0\rangle-\sqrt{\tfrac{2}{3}}\beta|1\rangle\right)|10\rangle+\tfrac{1}{\sqrt{6}}\beta|0\rangle|11\rangle\ . \end{aligned} \tag{53}$$

Bob and Charlie measure their system in the basis $\{0,1\}$ and communicate their result to Alice. If the two outcomes are different, then Alice can restore the original input with some probability by application of a filter operation. If Bob has found "0" and Charlie has found "1", Alices successful filter operation is described by the Kraus operator $A_F=\frac{1}{2}|0\rangle\langle 0|+|1\rangle\langle 1|$. If the results of Bob and Charlie are interchanged, Alice applies a filtering operation such that the success is described by $A_F=-|0\rangle\langle 0|-\frac{1}{2}|1\rangle\langle 1|$. The total probability of success, including the measurements of Bob and Charlie and the filtering operation, is again 1/3.

Note that our restoration scheme is related to ideas of quantum secret sharing in the sense of [21]: only when the parties are cooperating they are able to fully restore the original. Each of them does have some knowledge about the state, though, and therefore the cloning output state does not correspond to secret sharing in the spirit of [22], where none of the subsystems contains any information.

V. SUMMARY AND DISCUSSION

In this paper we have studied various measurements on parts of the output of a $1\rightarrow 2$ universal cloning machine. We have shown their correspondence to effective measurements on the input qubit.

In particular, we were interested in sharp effective measurements, as they are the ones that maximise the information gain and fidelity of state estimation, and allow state exclusion. We have found that measurements on any two output subsystems can correspond to sharp effective measurements at the input, whereas measurements on one output only can not. A complete set of sharp effective measurements at the input, though, can only be implemented by measurements on the two clones.

We have also studied the possibility of restoring the original in one of the output qubits, after performing certain measurements on the other outputs, and communicating classically. In scenarios where a party holds two of the three outputs this teleportation-like scheme was shown to operate with probability one. In a scenario where each party holds only one output the restoration is successful with probability smaller than one.

We hope that these results may be a step towards answering the question whether approximate quantum cloning is a useful process in quantum information processing.

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